Comparison of methods for computing the response coefficients in stochastic finite element analysis

Marc Berveiller (1), (2)  
Dr. Bruno Sudret (1)  
Pr. Maurice Lemaire (2)  

(1) Electricité de France – R&D Division, Les Renardières, 77818 Moret-sur-Loing - FRANCE  
(2) Institut Français de Mécanique Avancée, LaRAMA, Campus de Clermont-Ferrand, Les Cézeaux, BP265, 63175 Aubière - FRANCE

Abstract

This paper presents various methods for computing response coefficients in stochastic finite element analysis. The first one is a generalisation of the Stochastic Finite Element Method (SFEM) by Ghanem and Spanos (1991) in the case of random variables, which take into account randomness in material properties and loading. The second method uses the properties of orthogonality of the polynomial chaos; these coefficients may alternatively be cast as an expectation formula, which reduces to computing an integral. This computation may be carried out by a quadrature method. These methods are applied to the problem of the threshold of a tunnel in an elastic soil mass.

Keywords  
Finite element reliability; stochastic finite element; polynomial chaos; quadrature scheme.

Introduction

The Spectral Stochastic Finite Element Method (SSFEM) was introduced in the early 90’s for solving stochastic boundary value problems in which the spatial variability of a material property was modelled as a random field usually Gaussian or lognormal (Ghanem and Spanos (1991)). The aim of this paper is to extend this method to the case of any number of random variables of any type, which represents random onto material properties (e.g. Young’s modulus and/or Poisson’s ratio) and loading. The proposed stochastic finite element procedure (SFEP) relies upon two ingredients:

- expansion of the input random variables in terms of Hermite polynomials of standard normal variables,
- expansion of the response onto the so-called polynomial chaos.

Another method based on the quadrature method and the orthogonality of the polynomial chaos, allows to compute this response expansion with a series of deterministic finite element analysis.

Expansion of random variables onto the polynomial chaos

Let us denote by \( L^2(\Theta, F, P) \) the Hilbert space of random variables with finite variance. Let us consider a random variable \( X \) with prescribed probability density function (PDF) \( f_X(x) \). Classical results (Malliavin (1997)) allow to expand \( X \) as Hermite polynomial series expansion:
\[ X = \sum_{j=0}^{\infty} a_j H_j(\xi) \]  

(1)

where \( \xi \) denotes a standard normal variable, \( \{H_j, i = 0, \ldots, \infty\} \) are Hermite polynomials defined by:

\[ H_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} \left( e^{-\frac{x^2}{2}} \right) \]

(2)

and \( \{a_j, i = 0, \ldots, \infty\} \) are coefficients to be evaluated. Two methods are now presented for this purpose.

**Projection method**

This method was used by Puig et al. (2002), Xiu and Karniadakis (2002), Field and Grigoriu (2004). Due to the orthogonality of the Hermite polynomials with respect to the Gaussian measure, it comes from Eq. (1):

\[ \mathbb{E}[X H_j(\xi)] = a_j \mathbb{E}[H_j^2(\xi)] \quad , \quad \mathbb{E}[\cdot] \equiv \text{expectation} \]

(3)

where \( \mathbb{E}[H_j^2(\xi)] = j! \). Let \( F_X \) denote the cumulative distribution (CDF) of the random variable to approximate and \( \Phi \) the standard normal CDF. By using the transformation to the standard space \( X \rightarrow \xi : F_X(X) = \Phi(\xi) \), one can write:

\[ X(\xi) = F_X^{-1}(\Phi(\xi)) \]

(4)

Thus:

\[ a_i = \frac{1}{j!} \mathbb{E}[X(\xi)H_i(\xi)] = \frac{1}{j!} \int_{\mathbb{R}} F_X^{-1}(\Phi(t))H_i(t)\varphi(t)dt \]

(5)

where \( \varphi \) is the standard normal PDF. When \( X \) is a normal, lognormal or uniform random variable, coefficients \( \{a_i, i = 0, \ldots, \infty\} \) can be evaluated analytically:

- For normal: \( X \equiv N(\mu, \sigma) \quad a_0 = \mu, a_i = \sigma, a_0 = 0 \) for \( i \geq 2 \)
- For lognormal: \( X \equiv \text{LN}(\lambda, \xi) \quad a_i = \frac{\xi^i}{i!} e^{\left( \frac{\lambda^2}{2} \right)} \) for \( i \geq 0 \)

\[ X \equiv \text{U}[a, b] \quad a_0 = \frac{a+b}{2}, a_2i = 0, a_{2i+1} = \frac{(-1)^{i+1}(a-b)}{2^{i+1}\sqrt{\pi}(2i+1)!} (2i-1) \]

For other types of distribution, the quadrature method may be used for evaluating the integral in Eq. (5) (Berveiller and Sudret (2004)).

**Collocation method**

This method was introduced by Webster and al., (1996) and Isukapalli, (1999). It is based on a least square minimization of the discrepancy between the input variable \( X \) and its truncated approximation \( \tilde{X} \):

\[ \tilde{X} = \sum_{i=0}^{p} a_i H_i(\xi) \]

(6)

Let \( \{\xi^{(1)}, \ldots, \xi^{(n)}\} \) be \( n \) outcomes of \( \xi \). From Eq. (4), we obtain \( n \) outcomes \( \{X^{(1)}, \ldots, X^{(n)}\} \). The least square method consists in minimizing the following quantity with respect to \( \{a_i, i = 0, \ldots, p\} \):

\[ \Delta X = \sum_{i=1}^{n} \left( X^{(i)} - \tilde{X}^{(i)} \right)^2 = \sum_{i=1}^{n} \left( X^{(i)} - \sum_{j=0}^{p} a_j H_j(\xi^{(i)}) \right)^2 \]

(7)
This leads to the following linear system yielding the expansion coefficients \( \{a_i\}, i = 0, \ldots, p \):

\[
\begin{bmatrix}
\sum_{i=1}^{n} H_0(\xi^{(i)}) H_0(\zeta^{(i)}) & \cdots & \sum_{i=1}^{n} H_0(\xi^{(i)}) H_p(\zeta^{(i)}) \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{n} H_p(\xi^{(i)}) H_0(\zeta^{(i)}) & \cdots & \sum_{i=1}^{n} H_p(\xi^{(i)}) H_p(\zeta^{(i)})
\end{bmatrix}
\begin{bmatrix}
\sum_{i=1}^{n} X^{(i)} H_0(\zeta^{(i)}) \\
\vdots \\
\sum_{i=1}^{n} X^{(i)} H_p(\zeta^{(i)})
\end{bmatrix}
= 
\begin{bmatrix}
a_0 \\
\vdots \\
a_p
\end{bmatrix}
\]  

(8)

Both methods give good results (Berveiller and Sudret (2004)). Generally speaking, third order of expansion (\( p = 3 \)) provides the best results.

**Stochastic Finite Element Procedure in linear elasticity**

Using classical notations (Zienkiewicz and Taylor (2000)), the finite element method for static problems in linear elasticity yields a linear system of size \( N_{dd} \times N_{dd} \) where \( N_{dd} \) denotes the number of degrees of freedom of the structure:

\[
K \cdot U = F
\]

(9)

where \( K \) is the global stiffness matrix, \( U \) is the basic response quantity (vector of nodal displacement) and \( F \) is the vector of nodal forces.

In SFEP, due to the introduction of input random variables, the basic response quantity is a random vector of nodal displacements \( U(\theta) \). Each component is a random variable expanded onto the so-called polynomial chaos:

\[
U(\theta) = \sum_{j=0}^{\infty} U_j \psi_j \left( \left\{ \xi_k(\theta) \right\}_{k=1}^M \right)
\]

(10)

where \( \left\{ \xi_k(\theta) \right\}_{k=1}^M \) denotes the set of standard normal variables appearing in the discretization of all input random variables and \( \psi_j \left( \left\{ \xi_k(\theta) \right\}_{k=1}^M \right) \) are multidimensional Hermite polynomials that form an orthogonal basis of \( L^2(\Theta, F, P) \). As the polynomial chaos contains the one-dimensional Hermite polynomials in each random variable, the expansion of each input random variable Eq.(1) may be injected in the polynomial chaos expression:

\[
X^j = \sum_{j=0}^{\infty} x_j^j H_j(\xi_j) = \sum_{j=0}^{\infty} \tilde{\xi}_j^j \psi_j \left( \left\{ \xi_k(\theta) \right\}_{k=1}^M \right)
\]

(11)

In the sequel, the dependency of \( \psi_j \) in \( \left\{ \xi_k(\theta) \right\}_{k=1}^M \) will be omitted for the sake of clarity.

**Taking into account randomness in material properties**

In the deterministic case, the global stiffness matrix reads:

\[
K = \bigcup_c \sum_{\mathcal{E}_c} \int_{\Omega_c} B^T \cdot D \cdot B \ d\Omega_c
\]

(12)

where \( B \) is the matrix that relates the components of strain to the element nodal displacement, \( D \) is the elasticity matrix and \( \bigcup_c \) is the assembly procedure over all elements. When materials properties are described by means of random variables, the elasticity matrix hence the global stiffness matrix becomes random. The latter may be expanded onto the polynomial chaos as follows:
\[ K = \sum_{j=0}^\infty K_j \psi_j \]  
(13)

where

\[ K_j = E[ K \psi_j ] = \int_{\Omega} B^T \cdot E[ D \psi_j ] \cdot B \, d\Omega \]  
(14)

Note that \( B \) is a deterministic matrix while \( D \) is random. In case of an isotropic elastic material with random independent Young’s modulus \( E \) and Poisson’s ratio \( \nu \), the latter may be written as (plane strain problems):

\[ D = E \left( \tilde{\lambda}(\nu) D_1 + 2\tilde{\mu}(\nu) D_2 \right) \]  
(15)

where

\[ \tilde{\lambda}(\nu) = \frac{\nu}{(1+\nu)(1-2\nu)} \quad \tilde{\mu}(\nu) = \frac{1}{2(1+\nu)} \]  
(16)

and

\[ D_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad D_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]  
(17)

Random Young’s modulus \( E \) is expanded as in Eq.(11). Functions of the Poisson’s ratio \( \{ \tilde{\lambda}(\nu), \tilde{\mu}(\nu) \} \) may be expanded in the same way, using either the projection or the collocation method:

\[ E = \sum_{j=0}^\infty e_j H_i(\xi_E) \quad \tilde{\lambda}(\nu) = \sum_{j=0}^\infty \tilde{\lambda}_j H_i(\xi_\nu) \quad \tilde{\mu}(\nu) = \sum_{j=0}^\infty \mu_j H_i(\xi_\nu) \]  
(18)

Note that the same standard normal variable \( \xi_\nu \) is used to expand both functions \( \tilde{\lambda}(\nu) \) and \( \tilde{\mu}(\nu) \). By substituting for Eq.(18) in (15), one finally gets:

\[ D = \sum_{i=0}^\infty \sum_{j=0}^\infty e_i \tilde{\lambda}_j H_i(\xi_E) H_j(\xi_\nu) D_1 + \sum_{i=0}^\infty \sum_{j=0}^\infty e_i \mu_j H_i(\xi_E) H_j(\xi_\nu) D_2 \]  
(19)

If the structure under consideration is made of several materials, the above procedure is applied using different random variables in each element group having the same material properties. Eq.(19) may then be substituted into (14) to get the stiffness matrix expansion (13).

**Taking into account randomness in loading**

The vector of nodal forces may be written:

\[ F = \sum_{i=1}^{N_q} q^i F_i \]  
(20)

where \( N_q \) is the number of load cases, \( \{ q^i \}_{i=1}^{N_q} \) denote random loading parameters and \( F_i \) “load pattern” vectors corresponding to a unit value of \( q^i \). Note that this formulation equally applies to pinpoint forces, pressure or initial stresses. Coefficients \( q^i \) can be expanded onto the polynomial chaos (see Eq. (11)):

\[ q^i = \sum_{j=0}^\infty \tilde{q}^i_j H_j(\xi_i) = \sum_{j=0}^\infty \tilde{q}^i_j \psi_j \]  
(21)

Finally the random vector of nodal forces reads:
\[
\sum_{r=0}^{N_r} \sum_{j=0}^{\infty} \tilde{q}_r \psi_j \tilde{F}_r = \sum_{j=0}^{\infty} \tilde{F}_j \psi_j \quad (22)
\]

**Global linear system**

By using (10), (13) and (22), the discretized stochastic balance equation reads:

\[
\left( \sum_{i=0}^{\infty} K_{i,\psi_i} \right) \cdot \left( \sum_{j=0}^{\infty} U_{j,\psi_j} \right) = \sum_{j=0}^{\infty} \tilde{F}_j \psi_j 
\quad (23)
\]

After a truncation of the series appearing in Eq. (23) at order \( P \), the residual in this equilibrium equation is:

\[
\varepsilon_P = \left( \sum_{i=0}^{P-1} K_{i,\psi_i} \right) \cdot \left( \sum_{j=0}^{P-1} U_{j,\psi_j} \right) - \sum_{j=0}^{P-1} \tilde{F}_j \psi_j
\quad (24)
\]

Coefficients \( \{U_0, \ldots, U_{P-1}\} \) are computed using the Galerkin method minimizing the residual defined above, which is equivalent to requiring that this residual be orthogonal to the space spanned by \( \{\psi_j\}_{j=0}^{p-1} \) (Ghanem and Spanos, 1991):

\[
E[\varepsilon_P \psi_k] = 0 \quad , \quad k = \{0, \ldots, P-1\}
\quad (25)
\]

This leads to the linear system:

\[
\begin{pmatrix}
K_{0,0} & \cdots & K_{0, P-1} \\
K_{1,0} & \cdots & K_{1, P-1} \\
\vdots & \ddots & \vdots \\
K_{P-1,0} & \cdots & K_{P-1, P-1}
\end{pmatrix} \cdot
\begin{pmatrix}
U_0 \\
U_1 \\
\vdots \\
U_{P-1}
\end{pmatrix} =
\begin{pmatrix}
\tilde{F}_0 \\
\tilde{F}_1 \\
\vdots \\
\tilde{F}_{P-1}
\end{pmatrix}
\quad (26)
\]

where \( K_{jk} = \sum_{i=0}^{P-1} K_{i, E[\psi_j \psi_k]} = \sum_{j=0}^{P-1} K_{ij} d_{ijk} \) (cf. Appendix for an computation of \( d_{ijk} \)).

This system can be solve directly (Ghanem and Kruger, 1996)) or using a hierarchical approach (Ghanem, 1999, 2000) after a direct resolution at a lower order.

**Non intrusive Method**

The principle of the non-intrusive method (Field et al. (2000), Field (2002), Ghiocel et Ghanem (2002), Keese et Matthies (2003)) is to compute coefficients \( U_j \) from Eq.(10) with a succession of deterministic analysis, which can be performed by any classical finite element code. This approach spares the assembling of a large linear system (26) and its resolution. By using the orthogonality of the basis on the Hilbert space \( L^2(\Theta, F, P) \), Eq.(10) yields:

\[
U_j = \frac{E[U_{j,\psi_j}]}{E[\psi_j^2]} \quad , \quad j = \{0, \ldots, P-1\}
\quad (27)
\]

The denominator is easy to compute (Sudret and Der Kiureghian (2000)). The numerator can be evaluated by different methods:

**Simulation method**

As an expectation-like expression, the numerator of Eq.(27) can be computed with a simulation method like the Monte-Carlo simulation or a Latin Hypercube simulation.

**Quadrature method**

The numerator of Eq.(27) can be rewritten as:

\[
E[U_{j,\psi_j}] = \int_{\Omega} U(x) \psi_j(x) \varphi_M(x) d\Omega
\quad (28)
\]
where \( \varphi_M(x) = (2\pi)^{-M/2} e^{-\frac{1}{2}x^t M^{-1} x} \) denotes the \( M \)-dimensional multi-normal PDF. The fundamental theorem of Gaussian quadrature states that given an integer \( q \), we can find a set of weights \( w_j \) and a set of points \( x_j \) that satisfy:

\[
\int_{\mathbb{R}^q} U(x) \varphi_j(x) dx \approx \sum_{j=1}^{q} w_j \prod_{i=1}^{q} U(x_i, \ldots, x_i) \varphi_j(x_i, \ldots, x_i) \quad (29)
\]

where \( U(x_j) \) are results of deterministic analysis. All coefficients are obtained after \( q^M \) deterministic analysis.

**Post-processing**

It is easy to show that any response quantity (e.g. strain or stress component) may be also expanded onto the polynomial chaos (Berveiller and Sudret, (2004)). Thus the mechanical response of the system \( S \) (i.e. the set of all nodal displacements, strain or stress components) may be written as:

\[
S = \sum_{j=0}^{P-1} S_j \psi_j \quad (30)
\]

**Statistical moments of the response**

From Eq. (30), all statistical moments of the response can be easily computed. The mean of a response quantity \( s = \sum_{j=0}^{P-1} s_j \psi_j \) is:

\[
E[s] = s_0 \quad (31)
\]

The variance of \( s \) is:

\[
\text{Var}[s] = \sigma_s^2 = \sum_{j=1}^{P-1} E[\psi_j]^2 \quad (32)
\]

The skewness and the kurtosis of \( s \) are:

\[
\delta_s = \frac{1}{\sigma_s^3} \sum_{j=1}^{P-1} \sum_{k=1}^{P-1} \sum_{l=1}^{P-1} E[\psi_j \psi_k \psi_l] s_j s_k s_l
\]

\[
\kappa_s = \frac{1}{\sigma_s^4} \sum_{j=1}^{P-1} \sum_{k=1}^{P-1} \sum_{l=1}^{P-1} \sum_{m=1}^{P-1} E[\psi_j \psi_k \psi_l \psi_m] s_j s_k s_l s_m \quad (33)
\]

**Reliability analysis**

In reliability analysis, the failure criterion of a structure is defined in terms of a limit state function \( g(X, S(X)) \), which may depend both on basic random variables \( X \) and response quantity \( S(X) \). When using the SFEP and the polynomial representation of the response (Eq. (30)), it is clear that any limit state function is analytical and defined in terms of standard normal variables:

\[
g(X, S(X)) = g \left( \{\xi_k\}_{k=1}^{M}, \sum_{j=0}^{P-1} S_j \psi_j \left( \{\xi_k\}_{k=1}^{M} \right) \right) \quad (34)
\]

Thus the reliability problem, which is already formulated in the standard normal space, may be solved by any method including Monte-Carlo simulation, FORM/SORM, Importance Sampling, etc. (Ditlevsen and Madsen, (1996)).
**Representation of response PDF**
The probability density function of the response can be computed by various methods including Monte-Carlo simulation and parametric FORM analysis (Sudret and Der Kiureghian (2002)).

**Application example**

**Position of the problem**
Let us consider a deep tunnel in an elastic, isotropic homogenous soil mass. Let us consider a homogenous initial stress field. The coefficient of earth pressure at rest is defined as $K_0 = \frac{\sigma_{xx}^0}{\sigma_{yy}^0}$. Parameters of geometry, material properties and loads are given in Tab. 1. Due to the symmetry of the problem, only a quarter of the problem is modelled by finite element (Fig. 1). The mesh contains 462 nodes and 420 4-node linear elements, which allow a 1.4% accuracy evaluation of the radial settlement compared to a reference solution. The analysis is carried out under plane strain conditions.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Notation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tunnel depth</td>
<td>$L$</td>
<td>20 m</td>
</tr>
<tr>
<td>Tunnel radius</td>
<td>$R$</td>
<td>1 m</td>
</tr>
<tr>
<td>Vertical initial stress</td>
<td>$\sigma_{yy}^0$</td>
<td>-0.2 MPa</td>
</tr>
<tr>
<td>Coefficient of earth pressure at rest</td>
<td>$K_0$</td>
<td>0.5</td>
</tr>
<tr>
<td>Young modulus</td>
<td>$E$</td>
<td>50 MPa</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu$</td>
<td>0.2</td>
</tr>
</tbody>
</table>

**Tab. 1. Parameters of the deterministic model**

In the sequel, 4 parameters are considered as random. They are gathered in Tab. 2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Notation</th>
<th>Type</th>
<th>Mean</th>
<th>Coeff. Of Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical initial stress</td>
<td>$-\sigma_{yy}^0$</td>
<td>Lognormal</td>
<td>0.2 MPa</td>
<td>30%</td>
</tr>
<tr>
<td>Coefficient of earth pressure at rest</td>
<td>$K_0$</td>
<td>Lognormal</td>
<td>0.5</td>
<td>10%</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E$</td>
<td>Lognormal</td>
<td>50 MPa</td>
<td>20%</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu$</td>
<td>Uniform</td>
<td>0.2</td>
<td>29%</td>
</tr>
</tbody>
</table>

**Fig. 1. Geometry and mesh of the tunnel**

In the sequel, 4 parameters are considered as random. They are gathered in Tab. 2.
The maximum radial displacement of the tunnel is obtained at point E (Fig.1). Poulos and Davis, (1974) give the analytical solution :

\[ u_E = \frac{1+\nu}{E} \sigma_{yy} \left[ 2 - 2\nu - K_0 + 2\nu K_0 \right] \]  

(35)

**Statistical results**

The value of $u_E$ obtained for the mean values of the random parameters is $u_E^m = -0.00624$m. Statistical moments of $u_E$ are given in Tab. 3. Reference solutions are obtained by Monte Carlo simulation with 100000 samples (column #2) in Eq.(35). The SFEP method gives better results for moments of order 3 and 4 with the direct resolution (column #3) than with the hierarchical method (column #4). We can note that the quadrature method gives identical results whatever the number of integration points, those results are better than the SFEP for evaluating the skewness and the kurtosis.

<table>
<thead>
<tr>
<th>Tab. 3. Statistical moments of the settlement of the tunnel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>-----------------------------</td>
</tr>
<tr>
<td>$u_E / u_E^m$</td>
</tr>
<tr>
<td>Coeff. Of Variation</td>
</tr>
<tr>
<td>Skewness</td>
</tr>
<tr>
<td>Kurtosis</td>
</tr>
</tbody>
</table>

Tab. 4 reports the computer processing time for the complete resolution in each case (unit time corresponds to one single deterministic analysis). We can note that for the SFEP, which hierarchical solving is ten times faster than the direct solving. In the quadrature method, the CPU increases exponentially with the number of integration points (for $m$ variables and $n$ points, we need $n^m$ analysis).

<table>
<thead>
<tr>
<th>Tab. 4. Computer processing time required by the SFEP and the quadrature method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>-----------------------------</td>
</tr>
<tr>
<td>Time</td>
</tr>
</tbody>
</table>

Fig. 2 presents the PDF of the maximal settlement of the tunnel obtained by the various methods. This result is confirmed by values of coefficients of the radial displacement of the tunnel: there are equal for the three solving strategies.

**Conclusion**

This paper shows that there are two methods for computing the response coefficients in stochastic finite element analysis: the classical stochastic finite element procedure and a quadrature method, which allow to compute those coefficients only with a succession of deterministic analysis. Thus this method can be used for other material behaviour especially in a non-linear case. The accuracy of the approaches is equivalent and the quadrature method appears faster than the direct solving scheme in the application example.
Fig. 2. PDF of the maximum radial displacement

References


**Appendix**

The expectation of a product of three polynomials of gaussian random variables $D_{ijk} = \text{E}[H_i(\xi)H_j(\xi)H_k(\xi)]$ is:

$$
D_{ijk} = \begin{cases} 
\frac{i!j!k!}{(i+j-k)! (j+k-i)! (k+i-j)!}, & \text{if} \quad (i+j+k) \text{ even} \\
0, & \text{otherwise}
\end{cases}
$$

(36)

The $M$-th dimensional $p$-th order polynomial chaos consists in a set of multidimensional Hermite polynomials in $\{\xi_i\}_{i=1}^M$, whose degree does not exceed $p$. Each polynomial is completely defined by a sequence of $M$ non-negative integers $\alpha = \{\alpha_1, \ldots, \alpha_M\}$:

$$
\begin{align*}
\psi_i &= \prod_{m=1}^{M} H_{\alpha_m}(\xi_m), & \alpha_m &\geq 0 \\
\psi_j &= \prod_{m=1}^{M} H_{\beta_m}(\xi_m), & \beta_m &\geq 0 \\
\psi_k &= \prod_{m=1}^{M} H_{\gamma_m}(\xi_m), & \gamma_m &\geq 0
\end{align*}
$$

(37)

With Eqs.(36) and (37), one becomes:

$$
d_{ijk} = \text{E}[\psi_i\psi_j\psi_k] = \prod_{m=1}^{M} D_{\alpha_m\beta_m\gamma_m}
$$

(38)